



## 5.3 Inner products on other vector spaces

Douglas Wilhelm Harder, LEL, M.Math.

[dwharder@uwaterloo.ca](mailto:dwharder@uwaterloo.ca)

[dwharder@gmail.com](mailto:dwharder@gmail.com)





# Introduction

- In this topic, we will
  - Consider inner products on discrete-time signals
  - Consider inner products on continuous-time signals





# Discrete-time signals

- For discrete-time signals, if two such signals  $x$  and  $y$  both have finite 2-norms; that is,

$$\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} |x[k]|^2} < \infty \text{ and } \|y\|_2 = \sqrt{\sum_{k=0}^{\infty} |y[k]|^2} < \infty$$

then the inner product

$$\langle x, y \rangle = \sum_{k=0}^{\infty} x[k]^* y[k] < \infty$$

- Of course, if the signals are real, then  $x[k]^* = x[k]$





# Discrete-time signals

- For example, these geometric sequences have finite norms

$$\left\| \frac{1}{2^n} \right\|_2 = \sqrt{\frac{1}{1 - \frac{1}{4}}} = \frac{2}{\sqrt{3}} \quad \text{and} \quad \left\| \left( -\frac{1}{3} \right)^n \right\|_2 = \sqrt{\frac{1}{1 - \frac{1}{9}}} = \frac{3}{2\sqrt{2}}$$

The inner product of these two discrete-time signals is

$$\begin{aligned} \left\langle \frac{1}{2^n}, \left( -\frac{1}{3} \right)^n \right\rangle &= 1 - \frac{1}{6} + \frac{1}{36} - \frac{1}{216} + \frac{1}{1296} - \frac{1}{7776} + \dots \\ &= \frac{1}{1 - \left( -\frac{1}{6} \right)} = \frac{6}{7} \end{aligned}$$





# Discrete-time signals

- Note that the inner product of two geometric sequences with ratios  $\gamma$  and  $\delta$  is a geometric series with ratio  $\gamma\delta$

- If  $|\gamma| < 1$  and  $|\delta| < 1$  then the geometric series equals

$$\left\langle \gamma^n, \delta^n \right\rangle = 1 + \gamma\delta + (\gamma\delta)^2 + (\gamma\delta)^3 + \dots$$

$$= \frac{1}{1 - \gamma\delta}$$

- Note that under these conditions,  
the inner product is never zero





# Continuous-time signals

- Given an interval  $[a, b]$ , a 2-norm of a function on this interval is given by

$$\|f\|_2 = \sqrt{\int_a^b |f(t)|^2 dt}$$

- If this is finite for two functions  $f$  and  $g$ , we may define

$$\langle f, g \rangle = \int_a^b f(t)^* g(t) dt$$

- For example, if the interval is  $[0, 2\pi]$ , we have

$$\begin{aligned} \langle e^{j2t}, e^{j5t} \rangle &= \int_0^\pi (e^{j2t})^* (e^{j5t}) dt \\ &= \int_0^\pi e^{-j2t} e^{j5t} dt = \int_0^\pi e^{j3t} dt = 0 \end{aligned}$$





# Continuous-time signals

- Given an interval  $[0, \infty)$ , a 2-norm of a function on this interval is given by

$$\|f\|_2 = \sqrt{\int_0^{\infty} (f(t))^2 dt}$$

- If this is finite for two functions  $f$  and  $g$ , we may define

$$\langle f, g \rangle = \int_0^{\infty} f(t)^* g(t) dt$$

- In calculus, you will find that if  $\alpha > 0$  and  $\beta > 0$ ,

$$\begin{aligned} \langle e^{-\alpha t} \sin(t), e^{-\beta t} \cos(t) \rangle &= \int_0^{\infty} e^{-\alpha t} \sin(t) e^{-\beta t} \cos(t) dt \\ &= \frac{1}{(\alpha + \beta)^2 + 4} \end{aligned}$$





# The 2-norm induced by the inner product

- In each of the examples we discussed,  
note that the 2-norm is the induced norm

$$\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} |x[k]|^2} = \sqrt{\sum_{k=0}^{\infty} x[k]^* x[k]} = \sqrt{\langle x, x \rangle}$$

$$\|f\|_2 = \sqrt{\int_a^b |f(t)|^2 dt} = \sqrt{\int_a^b f(t)^* f(t) dt} = \sqrt{\langle f, f \rangle}$$

$$\|f\|_2 = \sqrt{\int_0^{\infty} |f(t)|^2 dt} = \sqrt{\int_0^{\infty} f(t)^* f(t) dt} = \sqrt{\langle f, f \rangle}$$





# Summary

- Following this topic, you now
  - Know that inner products can be defined for both
    - Discrete-time signals
    - Continuous-time signals
  - Understand we can find inner products if both operands have finite 2-norms
  - Are aware that the 2-norms we previously defined on these vector spaces is indeed the induced norm of these inner products





# References

- [1] [https://en.wikipedia.org/wiki/Norm\\_\(mathematics\)](https://en.wikipedia.org/wiki/Norm_(mathematics))





# Acknowledgments

None so far.





# Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





# Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.

